UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1 **May/June 2008**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1 The finite region enclosed by the line $y = kx$, where k is a positive constant, the x-axis for $0 \leqslant x \leqslant h$, and the line $x = h$ is rotated through 1 complete revolution about the x-axis. Prove by integration that the centroid of the resulting cone is at a distance $\frac{3}{4}h$ from the origin O. [4]

[The volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.]

2 Given that

$$u_n = \ln\left(\frac{1 + x^{n+1}}{1 + x^n}\right),$$

where $x > -1$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

Find the sum to infinity of the series

$$u_1 + u_2 + u_3 + \dots$$

when

(i) $-1 < x < 1$, [1]

(ii) $x = 1$. [1]

3 Show that if λ is an eigenvalue of the square matrix $\mathbf{A}$ with $\mathbf{e}$ as a corresponding eigenvector, and μ is an eigenvalue of the square matrix $\mathbf{B}$ for which $\mathbf{e}$ is also a corresponding eigenvector, then $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A} + \mathbf{B}$ with $\mathbf{e}$ as a corresponding eigenvector. [2]

The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -4 & -6 & -6 \\ 5 & 11 & 10 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as an eigenvector. Find the corresponding eigenvalue. [1]

The other two eigenvalues of $\mathbf{A}$ are 1 and 2, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively. The matrix $\mathbf{B}$ has eigenvalues 2, 3, 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ respectively. Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + \mathbf{B})^4 = \mathbf{PDP}^{-1}$. [3]

[You are not required to evaluate $\mathbf{P}^{-1}$.]

4 The curves C_1 and C_2 have polar equations

$$r = \theta + 2 \quad \text{and} \quad r = \theta^2$$

respectively, where $0 \leqslant \theta \leqslant \pi$.

(i) Find the polar coordinates of the point of intersection of C_1 and C_2. [2]

(ii) Sketch C_1 and C_2 on the same diagram. [2]

(iii) Find the area bounded by C_1, C_2 and the line $\theta = 0$. [3]

5 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $\alpha^3, \beta^3, \gamma^3$ is

$$y^3 - 3y^2 + 4y - 1 = 0.$$ [4]

Hence find the value of $\alpha^6 + \beta^6 + \gamma^6$. [3]

6 The curve C is defined parametrically by

$$x = 4t - t^2 \quad \text{and} \quad y = 1 - e^{-t},$$

where $0 \leqslant t < 2$. Show that at all points of C,

$$\frac{d^2y}{dx^2} = \frac{(t-1)e^{-t}}{4(2-t)^3}.$$ [4]

Show that the mean value of $\frac{d^2y}{dx^2}$ with respect to x over the interval $0 \leqslant x \leqslant \frac{7}{4}$ is

$$\frac{4e^{-\frac{1}{2}} - 3}{21}.$$ [4]

7 Prove by induction that

$$\sum_{r=1}^{n} \left(3r^5 + r^3\right) = \tfrac{1}{2}n^3(n+1)^3,$$

for all $n \geqslant 1$. [5]

Use this result together with the List of Formulae (MF10) to prove that

$$\sum_{r=1}^{n} r^5 = \tfrac{1}{12}n^2(n+1)^2 Q(n),$$

where $Q(n)$ is a quadratic function of n which is to be determined. [3]

8 **(i)** Given that

$$I_n = \int_0^{\frac{1}{2}\pi} t^n \sin t \, dt,$$

show that, for $n \geqslant 2$,

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}. \qquad [5]$$

(ii) A curve C in the x-y plane is defined parametrically in terms of t. It is given that

$$\frac{dx}{dt} = t^4(1 - \cos 2t) \quad \text{and} \quad \frac{dy}{dt} = t^4 \sin 2t.$$

Find the length of the arc of C from the point where $t = 0$ to the point where $t = \frac{1}{2}\pi$. [5]

9 The curve C has equation

$$y = \frac{x^2 - 2x + \lambda}{x + 1},$$

where λ is a constant. Show that the equations of the asymptotes of C are independent of λ. [3]

Find the value of λ for which the x-axis is a tangent to C, and sketch C in this case. [4]

Sketch C in the case $\lambda = -4$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

10 By considering $\sum_{n=1}^{N} z^{2n-1}$, where $z = e^{i\theta}$, show that

$$\sum_{n=1}^{N} \cos(2n-1)\theta = \frac{\sin(2N\theta)}{2\sin\theta},$$

where $\sin\theta \neq 0$. [6]

Deduce that

$$\sum_{n=1}^{N} (2n-1)\sin\left[\frac{(2n-1)\pi}{N}\right] = -N \operatorname{cosec}\frac{\pi}{N}. \qquad [4]$$

11 Show that, with a suitable value of the constant α, the substitution $y = x^\alpha w$ reduces the differential equation

$$2x^2\frac{d^2y}{dx^2} + (3x^2 + 8x)\frac{dy}{dx} + (x^2 + 6x + 4)y = f(x)$$

to

$$2\frac{d^2w}{dx^2} + 3\frac{dw}{dx} + w = f(x). \qquad [5]$$

Find the general solution for y in the case where $f(x) = 6\sin 2x + 7\cos 2x$. [6]

12 Answer only **one** of the following two alternatives.

EITHER

The position vectors of the points A, B, C, D are

$$7\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \qquad 3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}, \qquad 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}, \qquad 2\mathbf{i} + 7\mathbf{j} + \lambda\mathbf{k}$$

respectively. It is given that the shortest distance between the line AB and the line CD is 3.

(i) Show that $\lambda^2 - 5\lambda + 4 = 0$. [7]

(ii) Find the acute angle between the planes through A, B, D corresponding to the values of λ satisfying the equation in part **(i)**. [7]

OR

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\begin{pmatrix} 1 & 2 & -1 & -1 \\ 1 & 3 & -1 & 0 \\ 1 & 0 & 3 & 1 \\ 0 & 3 & -4 & -1 \end{pmatrix}.$$

The range space of T is denoted by V.

(i) Determine the dimension of V. [3]

(ii) Show that the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 3 \\ 0 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -1 \\ 3 \\ -4 \end{pmatrix}$ are linearly independent. [4]

(iii) Write down a basis of V. [1]

The set of elements of $\mathbb{R}^4$ which do not belong to V is denoted by W.

(iv) State, with a reason, whether W is a vector space. [1]

(v) Show that if the vector $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ belongs to W then $y - z - t \neq 0$. [5]

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